

Lecture 12 - Conductors

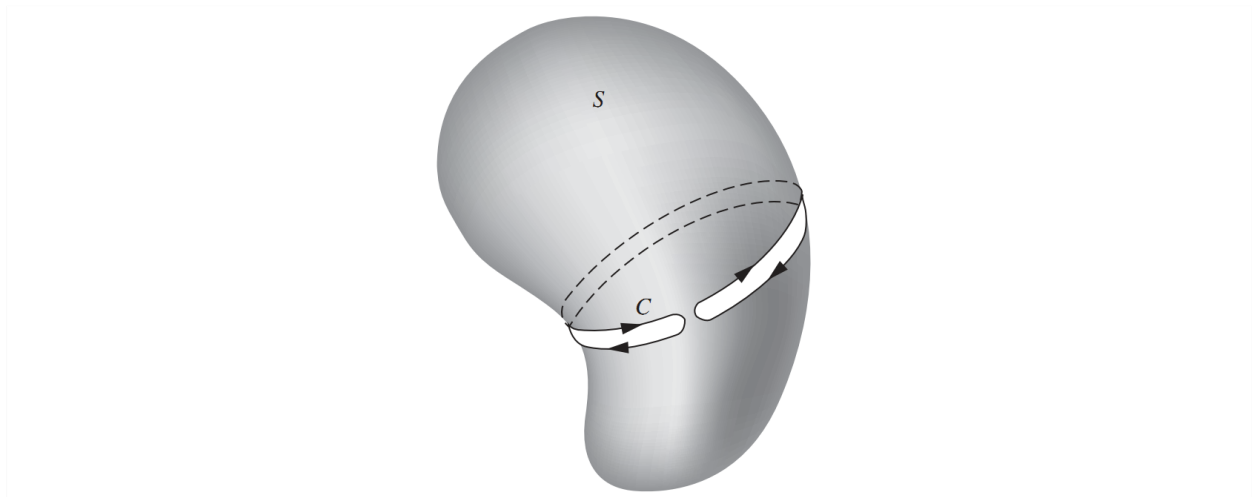
A Puzzle...

Divergence of the Curl

Prove that for any vector field \vec{A} with continuous derivatives, $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$.

Hint: Consider the surface S shown below bounded by the curve C . Consider the line integral around C and invoke Stoke's Theorem and Gauss's Theorem with suitable arguments. Recall that these two theorems are

$$\begin{aligned} \int_{\text{curve}} \vec{A} \cdot d\vec{s} &= \int_{\text{surface}} (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} \\ \int_{\text{surface}} \vec{F} \cdot d\vec{a} &= \int_{\text{volume}} \vec{\nabla} \cdot \vec{F} dV \end{aligned} \quad (1)$$



Solution

Of course, we could compute the divergence of a curl directly. For example, in Cartesian coordinates, $\vec{A} = \langle A_x, A_y, A_z \rangle$ and we find

$$\begin{aligned} \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right\rangle \\ &= \frac{\partial}{\partial x} \frac{\partial A_z}{\partial y} - \frac{\partial}{\partial x} \frac{\partial A_y}{\partial z} + \frac{\partial}{\partial y} \frac{\partial A_x}{\partial z} - \frac{\partial}{\partial y} \frac{\partial A_z}{\partial x} + \frac{\partial}{\partial z} \frac{\partial A_y}{\partial x} - \frac{\partial}{\partial z} \frac{\partial A_x}{\partial y} \\ &= \left(\frac{\partial}{\partial y} \frac{\partial A_x}{\partial z} - \frac{\partial}{\partial z} \frac{\partial A_x}{\partial y} \right) + \left(\frac{\partial}{\partial z} \frac{\partial A_y}{\partial x} - \frac{\partial}{\partial x} \frac{\partial A_y}{\partial z} \right) + \left(\frac{\partial}{\partial x} \frac{\partial A_z}{\partial y} - \frac{\partial}{\partial y} \frac{\partial A_z}{\partial x} \right) \\ &= 0 \end{aligned} \quad (2)$$

where the last step follows from the fact that the partial derivative operation commutes (i.e. taking $\frac{\partial}{\partial x} \frac{\partial}{\partial y}$ is the same as taking $\frac{\partial}{\partial y} \frac{\partial}{\partial x}$) for any function with continuous derivatives.

However, we can be much more inspired and actually follow the hint in the problem. The idea is very simply to make C infinitesimally thin so that the curve C becomes a line that goes forward and then back along itself. In this limit, the surface S becomes a closed surface, and so we find

$$\int_C \vec{A} \cdot d\vec{s} = \int_S (\nabla \times \vec{A}) \cdot d\vec{a} = \int_V \nabla \cdot (\nabla \times \vec{A}) dv \tag{3}$$

where V is the volume enclosed by S . Because C goes back along itself, the line integral

$\int_C \vec{A} \cdot d\vec{s} = 0 = \int_V \nabla \cdot (\nabla \times \vec{A}) dv$. This analysis can be repeated on any infinitesimal volume in space, which proves that

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \tag{4}$$

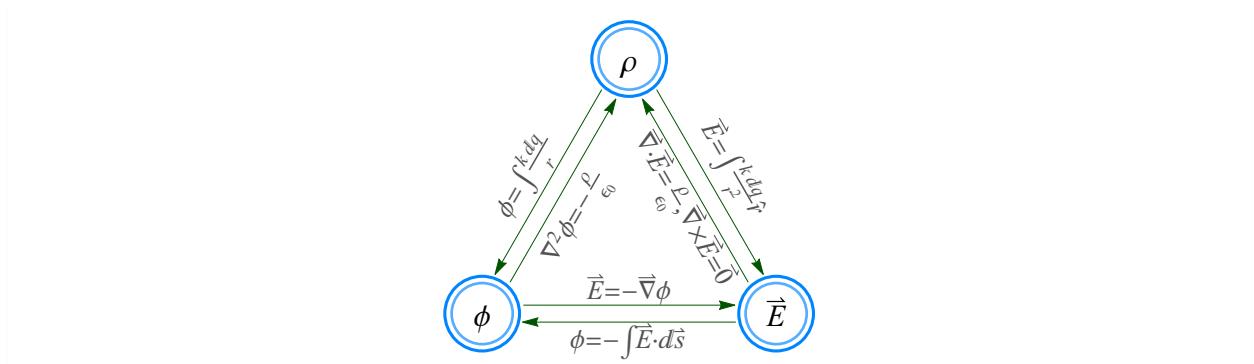
everywhere in space (for if $\nabla \cdot (\nabla \times \vec{A}) \neq 0$ at some point, we could pick a volume small enough around that point so that $\nabla \cdot (\nabla \times \vec{A})$ is essentially constant, so that there would be no possibility of any cancellation happening, and then we would reach a contradiction).

This result essentially boils down to the fact that the boundary of a boundary is zero. In other words, the boundary S of V has a boundary C which doesn't exist (you can either make C a line that goes back on itself as we did above, or simply make it a line of zero length). \square

A 30,000 Foot View

It is important to realize that although we have been focusing specific charge distributions from points, lines, sheets, and spheres, electricity is one of the most prevalent and influential forces that we encounter in our daily lives. Often times, we do not appreciate the fact that nearly every action we take - from using our five senses to the interactions between bubbles to the mechanism underlying printers to the fact that we do not fall through the ground - is connected to electricity. For example, this remarkable [YouTube video](#) highlights some amazing electricity phenomena that you can easily test out for yourself!

So where are we in our study of electricity? Over the past few weeks, we have built up a lot of intuition about electrostatics, determining the electric field (\vec{E}) and electric potential (ϕ) from many different charge distributions (ρ). The relations between these three key quantities is summarized in the following diagram:



We now begin to investigate the electrical properties of different materials. After this, we will be ready to discuss the effects of moving charges. From there, we will be poised to combine everything that we have learned (both special relativity and electricity) to find the most spectacular result of your physics career...! Get pumped!!!

Theory

Conductors

A (perfect) conductor has enough free charge that any applied electric field will be (almost instantly) countered by charge redistributing itself on the conductor. We only consider the electrostatic case where the charge has redis-

tributed and everything has stopped moving. Of course, there is no guarantee that, given a conductor, the charge will ever stop moving. But if it does, then we know the following things:

1. $\vec{E} = \vec{0}$ inside a conductor (otherwise, charge would move until this electrostatics case is reached)
2. $\rho = 0$ inside a conductor (from Gauss's Law $\vec{\nabla} \cdot \vec{E} = 0 = \frac{\rho}{\epsilon_0}$)
3. Any net charge resides on the surface (that is the only place it can be)
4. A conductor's surface is equipotential (inside the conductor $\vec{E} = \vec{0}$ so that the potential is constant. If the potential was not also constant on the surface, then charge would redistribute itself on the surface to make it so)
5. \vec{E} is perpendicular to the surface, just outside a conductor (any tangential \vec{E} would drive charge around on the surface, breaking our assumption that the electrostatic case has been reached)

A Conducting Spherical Shell

From here on out, when we talk about sheets of charge (e.g. a thin sheet on the x - y plane or a spherical shell of charge), we are going to assume that the sheet has a non-zero (albeit very tiny) thickness. This distinction is important, since each conductor must be treated as having some "inside" region where the electric field $\vec{E} = \vec{0}$. In addition, this implies that we must specify the charges on both the inside and the outside of every sheet. The following example illustrates this point.

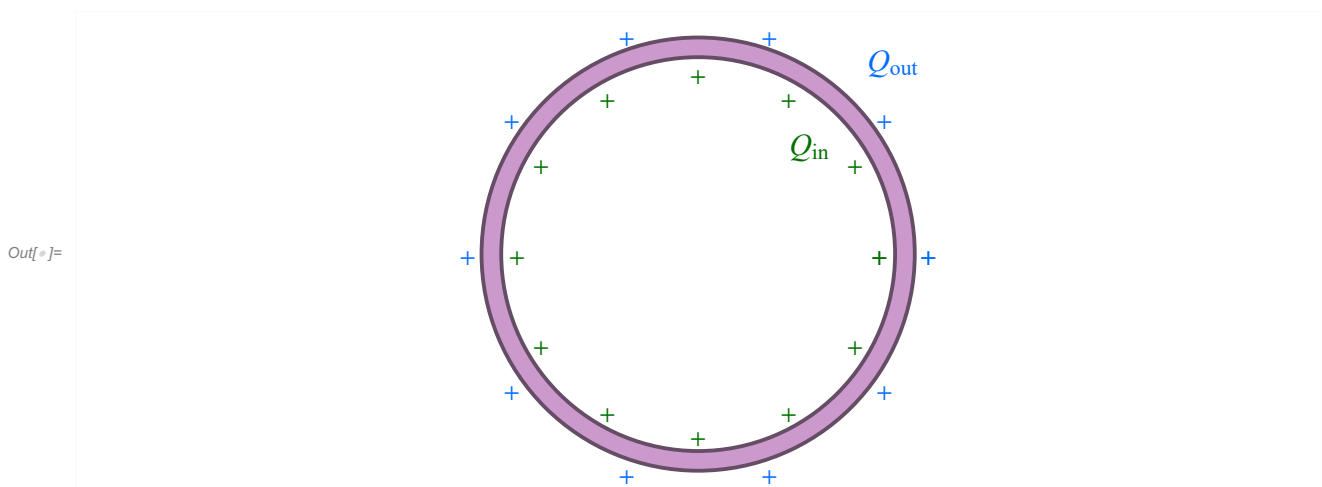
Example

Consider a spherical conducting shell with radius R and net charge Q . How will the charge distribute on the conductor?

Solution

Because the setup is spherically symmetric, the final electrostatic charge distribution will also be spherically symmetric. Note, however, that this need not be the case in general - a single surface of a conductor does not need to have uniform charge density.

Given that the charges will evenly distribute on the surfaces of the conductor, the only choice we have to make in this problem is how much total charge goes onto the outer surface Q_{out} and how much goes onto the inner surface Q_{in} .



Although we drew both Q_{out} and Q_{in} as positive charges above, either one can also be negative. The only constraint is that the total charge $Q_{\text{out}} + Q_{\text{in}} = Q$. Furthermore, although we drew the charges as distinct, they really

form a uniform charge density $\sigma_{\text{out}} = \frac{Q_{\text{out}}}{4\pi R^2}$ on the outside of the conductor and $\sigma_{\text{in}} = \frac{Q_{\text{in}}}{4\pi R^2}$ (where we have ignore the thickness of the inner and outer radii of the conductor).

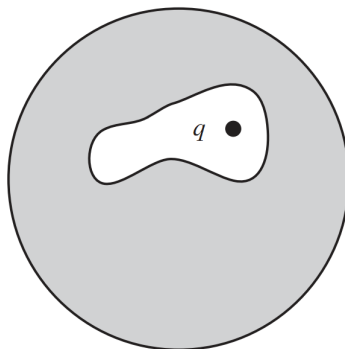
Using the Principle of Superposition, we can analyze each of the two spherical shells of charge independently. Recall that a shell of charge generates an electric field $\vec{E} = \frac{kq}{r^2} \hat{r}$ outside the shell and $\vec{E} = \vec{0}$ inside the shell.

Therefore, the outer shell of Q_{out} charge generates an electric field $\vec{E} = \vec{0}$ inside the conductor. On the other hand, the inner shell of Q_{in} charge generates an electric field $\vec{E} = \frac{kQ_{\text{in}}}{R^2} \hat{r}$ inside the conductor. By superposition, the net electric charge in the conductor will therefore be $\vec{E} = \frac{kQ_{\text{in}}}{R^2} \hat{r}$. Since we know that for an electrostatic situation, $\vec{E} = \vec{0}$ inside of a conductor, this forces $Q_{\text{in}} = 0$ and therefore $Q_{\text{out}} = Q$. Thus all of the charge Q will distribute itself evenly on the outside surface of a spherical conducting shell. \square

Complementary Section: Cavities Inside Conductors

Note that $\vec{E} = \vec{0}$ only applies to the meat of a conductor. If a neutral conductor has an inner cavity, then the electric field does not need to be zero in the cavity:

- If the cavity is empty of charge, then $\vec{E} = \vec{0}$ within the cavity.
- If there is charge q in the cavity, then a total charge $-q$ will redistribute itself along the inside wall of the cavity to attain zero electric field *everywhere in space*. For the neutral conductor, a charge q must distribute itself on the outside of a conductor in a way that satisfies the conditions listed above.



As an example, an uncharged *spherical* conductor with an arbitrary cavity in it which contains a charge q , then the walls of the cavity will redistribute $-q$ of charge to block the field of this point charge, and to keep the conductor neutral $+q$ charge will distribute itself *uniformly* on the surface of the conductor (uniformly because the charges repel each other and the influence of the q charge inside the cavity has been canceled out). Hence, the electric field outside the conductor will be $\frac{kq}{r^2} \hat{r}$. The nature of the cavity is hidden by the conductor, and the only information is the *net charge* inside all of its cavities. If the outside of the conductor was not spherical, the charge will not be uniformly spread on the surface, but rather would distribute itself so that the electric field was perpendicular to the surface of the conductor.

Grounding

You can ground a conductor by connecting it to an infinite charge reservoir which will ensure that the conductor has the same potential as infinity. For a finite charge distribution, this implies that the conductor has potential $\phi = 0$. For example, from the above section titled [A Conducting Spherical Shell](#), we saw that a total charge Q placed on a conducting sphere will rearrange itself so that the charge is uniformly spread over the outer surface of

the conductor. The potential at the surface of the conductor will then be $\phi = \frac{kQ}{R}$. If we then ground the conductor, what will happen? Charge will flow into or out of the system until $\phi = 0$, so in this case the charge Q will all flow out of the shell, leaving $\vec{E} = \vec{0}$ everywhere and therefore $\phi = 0$ everywhere.

Any time that an object is described as *metallic*, it implies that this object is a perfect conductor. The opposite of a conductor is an insulator (all the objects that we have been considering up until this point have been insulators). You can disperse charge on insulators however you want, and for a (perfect) insulator that charge will not move at all.

For example, if we take a sphere and uniformly distribute a charge Q over the outside of one hemisphere and a charge $-Q$ over the outside of the other hemisphere, the charges will stay there, and we can analyze the electric field as the superposition of two oppositely charge hemispheres. On the other hand, if you deposit this charge on a conductor, the charge will instantly rearrange so that the electric field within the conductor is zero. One such rearrangement is for the $+Q$ and $-Q$ charges to both distribute themselves uniformly on the surface of the sphere, which will generate no electric field anywhere (and hence it will definitely satisfy the requirement that there is no electric field inside the conductor). As we will find out in the next section, this solution is unique, and hence this is exactly what happens if you try and distribute charge over a hemisphere.

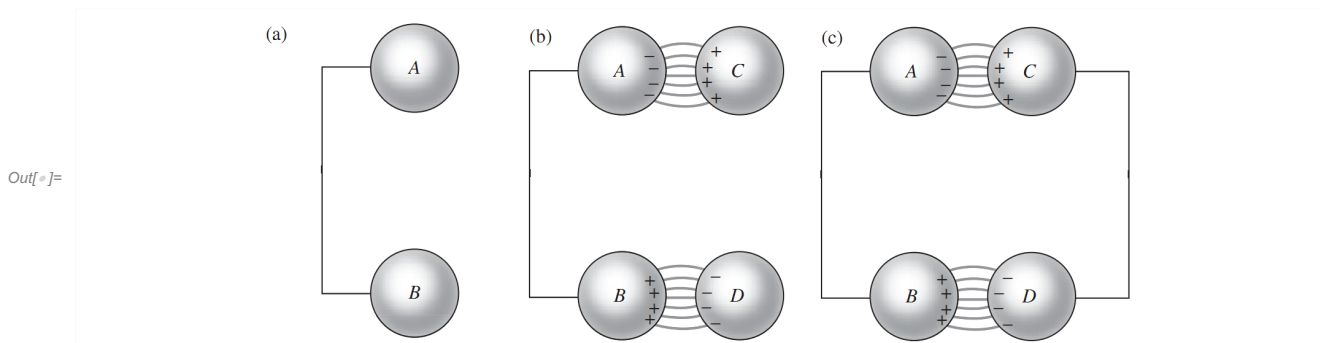
Uniqueness of Solutions

Consider a system of conductors, where the j^{th} conductor has a potential ϕ_j . The results found above are all based upon the *uniqueness theorem*,

Assuming that there is a solution $\varphi[x, y, z]$ for a given set of conductors
with given potentials φ_j (or charge Q_j), this solution must be unique. (5)

Example

The two metal spheres in are connected by a wire as shown in Panel (a); the total charge is zero. In Panel (b) two oppositely charged conducting spheres have been brought into the positions shown, inducing charges of opposite sign in A and in B . If now C and D are connected by a wire as in Panel (c), it could be argued that something like the charge distribution in Panel (b) ought to persist, each charge concentration being held in place by the attraction of the opposite charge nearby. Can this charge distribution remain?



Solution

Since the positive charges are very near negative charges (where they like to be) you might well guess that nothing will happen - the configuration still looks stable.

Well, that sounds reasonable, but it's wrong. The configuration shown in the above figure on the right is impossible. For there are now effectively two conductors, and the total charge on each is zero. One possible way to distribute zero charge over these conductors is to have no accumulation of charge anywhere, and hence zero field

everywhere. By the uniqueness theorem, this must be the solution: The charge will flow down the tiny wires, canceling itself off.

Another way to see that the configuration in Panel (c) can't be stable is to consider the path that runs from conductor B across the gap to conductor D , then through the interior of the wire that connects D to C , then across the gap to A , then finally via the other wire down to B . The line integral of \vec{E} around any closed path must be zero, if \vec{E} is a static electric field. But if the fields are as shown in Panel (c), the line integral over the closed path just described is not zero. Each gap makes a positive contribution; but in the conductors, including the connecting wires, $\vec{E} = \vec{0}$. So the proposed situation cannot represent a static charge distribution.

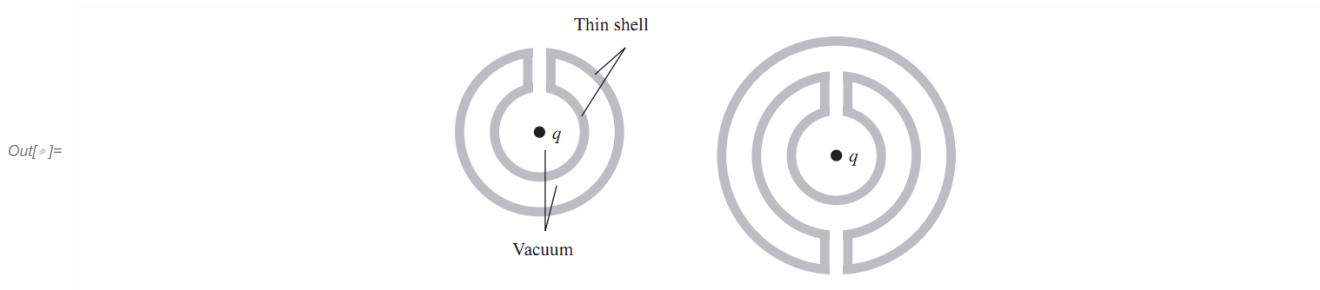
If you are looking for a more “cause and effect” reason why the charge redistributes itself, what happens is this: the charge that C induces on A (and likewise that D induces on B) isn't enough to keep all of C 's charge in place when the wire is connected. The self-repulsion of the charges within C wins out over the attraction from A . The quantitative details are contained (for the most part) in Problem 3.13 of Purcell and Morin. \square

Problems

In or Out

Example

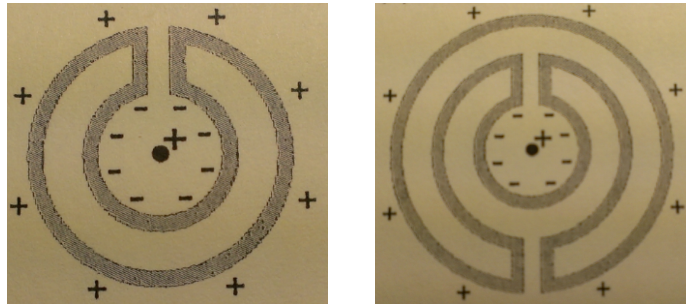
A positive charge q is placed at the center of each of the neutral spherical-ish hollow conducting shells whose cross sections are shown below (think about rotating the figures along the vertical axis to obtain the actual 3D setup). White areas on the page denote vacuum; the shaded curves denote the metal of the conducting shells. For each case, indicate roughly the induced charge distribution on the conductor. Be sure to indicate which part of the surface the charge lies on.



Solution

In both cases, the inner-most surface of the conductor will acquire a charge $-q$ spread evenly across its surface while the outer-most surface will acquire a charge $+q$ to keep the conductor neutral. This will ensure that the electric field inside of the conductor at any point will be zero, and the uniqueness theorem guarantees that it is the only solution.

Out[]:=



Notice that in the left figure, the charge is only induced along the outside of the conductor, consistent with the fact that if there is zero charge inside of a conductor's cavity, there will be zero charge along the inside surface of that cavity. In the second case, the charge is on the inside of the conductor, so there will be a corresponding charge $-q$ induced on the inside of the conductor to counter this point charge's effects. \square

Complementary Section: Grounding a Shell

Complementary Section: Zero Flow

Example

Recall the example “Dividing the Charge” from Lecture 11 where we minimized the energy of placing a total charge Q onto two conducting spherical shells of radius R_1 and R_2 placed far apart. Given a total amount of charge Q , we divide the charge up as $Q_1 = \frac{R_1}{R_1+R_2} Q$ and $Q_2 = \frac{R_2}{R_1+R_2} Q$ to minimize the potential energy of the system. We also found out that the potential on both spheres is the same; this means that if the two spheres were connected by a thin wire, then no charge would flow between them.

However, this last statement is a bit strange. Since the electric field is proportional to $\frac{1}{r^2}$, the field is much larger at the surface of the smaller sphere than the larger sphere, because the charge is proportional only to r . So why doesn't the charge get repelled from the smaller sphere and flow through the wire to the larger sphere?

Hint: Assume that only a tiny bit of charge flows onto this wire, and that this charge is uniformly distributed to make a charge density λ . What is the total force on this wire?

Solution

Assuming that the thin wire has constant charge density λ , the total force on it due to the sphere of radius R_1 equals

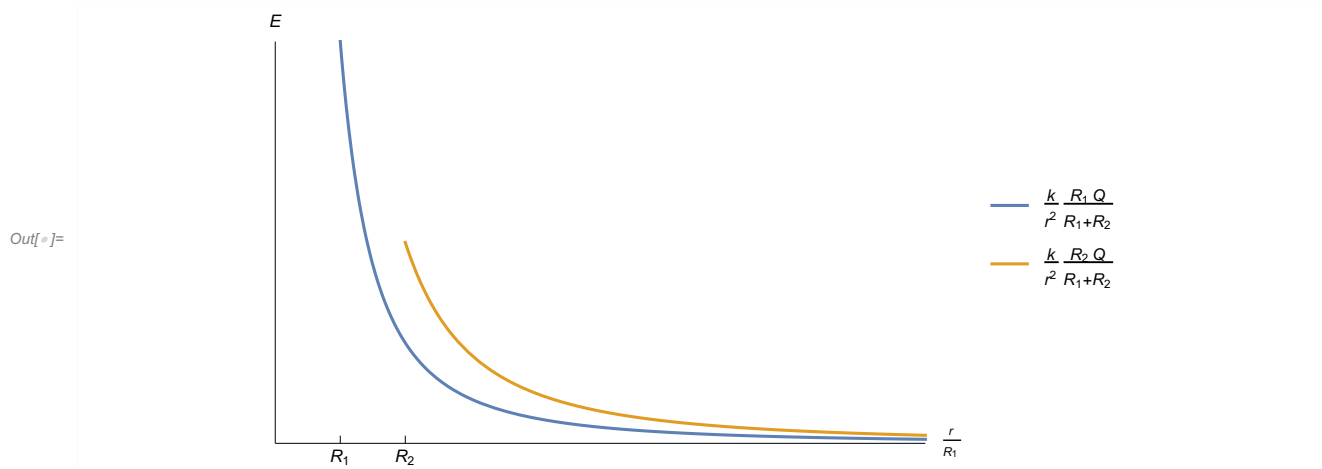
$$\int_{R_1}^{\infty} \frac{kQ_1}{r^2} dr = \frac{kQ_1}{R_1} = \frac{kQ}{R_1+R_2} \quad (10)$$

where we have placed the upper bound at infinity because we assumed that the two spheres are very far apart (more accurately, the integral would be $\int_{R_1}^{D-R_2} \frac{kQ_1}{r^2} dr$ where D is the distance between the centers of the sphere, but by making D large enough we can make the correction term between this integral and our integral negligible).

Similarly, the force due to the second sphere equals

$$\int_{R_2}^{\infty} \frac{kQ_2}{r^2} dr = \frac{kQ_2}{R_2} = \frac{kQ}{R_1+R_2} \quad (11)$$

Therefore, the force on the wire is equal, and the charges won't move between them. Note that although the force on the surface of the smaller sphere is greater, the force from the smaller sphere also dies off much faster than that of the larger sphere. The full integrals shows that they are ultimately equivalent, and this is also shown by the graph below.



So only a very tiny bit of charge goes onto the wire, and all the remaining charge will stay on the two spheres. The only remaining point is to discuss our initial assumptions: that only a tiny bit of charge goes onto the wire and that this charge is uniformly distributed along the wire. The first point is discussed in Problem 3.10 where it is shown that a thin wire has essentially zero capacitance. The second point is discussed in Problem 3.5, which is definitely worth a read! \square

Advanced Section: Electrostatic Pressure

Recommended Problems

This is a list of excellent problems (with solutions) in David Morin's book.

- 3.4 Charge distribution on a conducting disk - Guaranteed to blow your mind!
- 3.5 Charge distribution on a conducting stick

Mathematica Initialization